Appendix: Maximal power of heat machines

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When a motor generates work by exchanging heat with two sources at temperatures $T_1 > T_2$, Carnot's maximum efficiency $1 - T_2/T_1$ can be approached only for reversible processes; but these are too slow to yield significant power. At the other extreme, rapid processes are strongly irreversible and cannot produce much work. A compromise is needed to maximise the power output. Jacques Yvon tackled this problem in 1955, focusing on a major cause of irreversibility, the delay required for heat transport. A simple approach is presented here.

During the first step of a cycle, between the times t_0 and $t_1 = t_0 + \tau_1$, the motor M evolves at a temperature $\Theta_1(t)$, receiving the heat Q_1 from the hot source S_1 at temperature T_1. We assume that heat transfers obey Fourier's law, so that the heat flux $\Phi_1(t)$ from S_1 to M has at times t ($t_0 \le t \le t_1$) the form $\Phi_1(t) = C_1[T_1 - \Theta_1(t)]$ where C_1 is supposed to be given. During the second step, between t_1 and $t_2 = t_1 + \tau_\mathrm{A}$, M evolves adiabatically, its temperature going down from $\Theta_1(t_1)$ to $\Theta_2(t_2)$. During the third step, between t_2 and $t_3 = t_2 + \tau_2$, M evolves at a temperature $\Theta_2(t)$, giving the heat Q_1 to the cold source S_2 at temperature T_2, through a flux $\Phi_2(t) = C_2[\Theta_2(t) - T_2](t_2 \le t \le t_3)$. Finally, between t_3 and $t_4 = t_3 + \tau'_\mathrm{A}$, M returns adiabatically to its initial state, with a temperature rising from $\Theta_2(t_3)$ to $\Theta_1(t_4) = \Theta_1(t_0)$. The cycle has the total duration $\tau = t_4 - t_0 = \tau_1 + \tau_\mathrm{A} + \tau_2 + \tau'_\mathrm{A}$. The work produced by M during a cycle is the difference between the heat $Q_1 = \int_{t_0}^{t_1} \mathrm{d}t\, \Phi_1(t)$ that it received from S_1 and the heat $Q_2 = \int_{t_2}^{t_3} \mathrm{d}t\, \Phi_2(t)$ that it yielded to S_2, so that the average power output is

$$P = \frac{1}{\tau}\left\{C_1 \int_{t_0}^{t_1} \mathrm{d}t \left[T_1 - \Theta_1(t)\right] - C_2 \int_{t_2}^{t_3} \mathrm{d}t \left[\Theta_2(t) - T_2\right]\right\} . \tag{1}$$

The system M evolves in a closed cycle, so that its change ΔS of entropy between the times t_0 and t_4 vanishes:

$$\Delta S = C_1 \int_{t_0}^{t_1} \mathrm{d}t \frac{\left[T_1 - \Theta_1(t)\right]}{\Theta_1(t)} - C_2 \int_{t_2}^{t_3} \mathrm{d}t \frac{\left[\Theta_2(t) - T_2\right]}{\Theta_2(t)} = 0 . \tag{2}$$

We wish to maximise P as function of the temperatures $\Theta_1(t)$, $\Theta_2(t)$ and of the durations τ_1, τ_A, τ_2, τ'_A, for given values of T_1, T_2, C_1, C_2, τ, under the constraint $\Delta S = 0$. Introducing a Lagrange multiplier λ/τ, and writing that $P - \lambda\Delta S/\tau$ is stationary with respect to $\Theta_1(t)$ and $\Theta_2(t)$, we obtain

$$\Theta_1(t) = \sqrt{\lambda T_1}\ , \quad \Theta_2(t) = \sqrt{\lambda T_2}\ . \tag{3}$$

The optimal cycles for M are therefore Carnot cycles with constant temperatures $\Theta_1(t) \equiv \Theta_1$ and $\Theta_2(t) \equiv \Theta_2$. Maximising P with respect to τ_A and τ'_A provides $\tau_\mathrm{A}/\tau \simeq 0$, $\tau'_\mathrm{A}/\tau \simeq 0$: The adiabatic steps should be the shortest possible. It remains to write that $P - \lambda\Delta S/\tau$ is stationary with respect to $\tau_1/\tau = 1 - \tau_2/\tau$, which yields

$$C_1(T_1 - \Theta_1)(1 - \lambda/\Theta_1) + C_2(\Theta_2 - T_2)(1 - \lambda/\Theta_2) = 0\ , \tag{4}$$

or equivalently, using (3),

$$\sqrt{\lambda} = \frac{\sqrt{C_1 T_1} + \sqrt{C_2 T_2}}{\sqrt{C_1} + \sqrt{C_2}}\ . \tag{5}$$

From the condition $\Delta S = 0$, we get the optimal durations of the isothermal steps of the cycle:

$$\frac{\tau_1}{\tau} = \frac{\sqrt{C_2}}{\sqrt{C_1}+\sqrt{C_2}}, \quad \frac{\tau_2}{\tau} = \frac{\sqrt{C_1}}{\sqrt{C_1}+\sqrt{C_2}} \ . \tag{6}$$

Altogether, the maximal power is

$$P\max = \frac{C_1 C_2}{\left(\sqrt{C_1}+\sqrt{C_2}\right)^2}\,(T_1 - T_2) \ . \tag{7}$$

This power is equivalent to a fraction, at most equal to $\frac{1}{4}$ (reached for $C_1 = C_2$), of the heat flux $C(T_1 - T_2)$ that would be transferred from S_1 to S_2 for a coefficient C equal to the average $\sqrt{C_1 C_2}$. Finally, the efficiency at maximal power is obtained from $\Delta S = Q_1/\Theta_1 - Q_2/\Theta_2 = 0$ as

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\Theta_2}{\Theta_1} = 1 - \sqrt{\frac{T_2}{T_1}} \ . \tag{8}$$